HOW CAN ONE HEAR THE SHAPE OF A DRUM ?

Vu Kim Tuan
Department of Mathematics
University of West Georgia
Carrollton, GA 30118, USA

In 1966 Kac posed a question: *Can one hear the shape of a drum?* In other words, whether the shape of a domain can be determined by the eigenvalues of the Dirichlet Laplacian operator? It is an inverse spectral problem. It took almost thirty years to answer negatively Kac's question. In 1992, Gordon, Webb and Wolpert constructed two different drums which vibrate in exactly the same way. However, for a certain class of drums, a positive answer was proposed recently by Zelditch who showed that the problem has a unique solution in the class of drum surfaces with no holes and very smooth boundary and with at least one mirror symmetry.

In this talk we consider a more general problem. The inverse problem is to reconstruct both the shape of the unknown domain Ω and the unknown potential q associated with the Schrödinger equation

$$\begin{cases} -\Delta u + (\xi + q(x))\, u(x) = 1_\Omega(x)\psi(x), & x \in \Omega \subset \mathbb{R}^d, \quad d \geq 2, \\ u(x) = 0 \quad \text{for } x \in \partial\Omega. \end{cases}$$

Here $1_\Omega(x)$ denotes the characteristic function of Ω. We show that the values $u(b)$ of the solution at an arbitrary single point $b \in \Omega$ that are obtained by varying the frequency ξ over a (convergent) sequence and the source ψ over a certain basis are enough to construct the first eigenfunction. As the domain is not available, all methods relying on the Dirichlet-to-Neumann map and Calerman inequalities are not applicable. In our case, we collect the data by "listening" to the drum at one point and do not assume, that the sequence of eigenvalues is given as in Kac's problem. As in scattering theory, where the solution outside the domain satisfies some external condition as being the sum of the incident and reflective waves, here we assume that our solution vanishes outside Ω, and the source ψ plays the role of the incident wave as it will be defined on a larger domain. To overcome the missing crucial information about Ω, we provide a new reconstruction method that extract the spectral data from measurement at a single point, and so easier to implement and practical for engineering purposes. As the readings at a single point are generated by two sequences, the method lends itself to simple approximation by a finite number of measurements.

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