The boundary inversion of the heat equation

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Abstract. We are concerned with the recovery of the heat coefficient of a rectangular plate, $\Xi = [0,a] \times [0,b]$ say, from temperature readings taken at certain points on the lateral boundary. Thus if $u(x,y,t)$ denotes the temperature at a point $(x,y) \in \Xi$ and at time t, then it is solution of the heat equation

$$\left\{ \begin{array}{ll} \partial_t u(x,y,t) = \Delta u(x,y,t) - (q(x)+p(y))\, u(x,y,t), & 0<x<a,\ 0<y<b,\ \ t>0, \\ u(x,y,0) = f(x,y),\ \ u(x,0,t) = u(0,y,t) = u(x,b,t) = u(a,y,t) = 0, & \end{array} \right. \tag{0.1}$$

where $f(x,y)$ is the initial temperature of Ξ. For the sake of simplicity we assume that the heat coefficient decomposes into the sum of two one dimensional functions, i.e. $q(x)+p(y)$, which in turn simplifies the reading to be at three points only, namely corners of $\partial\Xi$. The usual way to solve this type of inverse problem is through the Dirichlet-to-Neumann map, or the boundary control method. There, the data involves the whole trace of the solution on the boundary. In this work we provide the reconstruction procedure and the underlying data processing as well. To this end, we shall read the gradient of temperature at three corners, i.e. we use the mapping

$$f(x,y) \overset{\Gamma}{\to} \{\partial_x\partial_y u(0,0,t), \partial_x\partial_y u(a,0,t),\ \partial_x\partial_y u(0,b,t)\} \qquad \text{for} \qquad 0<t<T\leq\infty. \tag{0.2}$$

In case the boundary lateral conditions in (0.1) are of Neumann type, then we would measure the temperatures at the corners

$$f(x,y) \overset{\Gamma}{\to} \{u(0,0,t), u(a,0,t), u(0,b,t)\}\,,$$

and the setting can be generalized to higher dimensions. Observe that the geometry of the rectangle Ξ allows for tangential and mixed derivatives of the data at the corners. As a simple application, one could imagine monitoring the heat dissipation or profile inside a micro-chip, from the measurement of the temperature at its corners. This setting can also easily be implemented, in reactors and engines, as it requires few temperature sensors, or thermometers, placed on the boundary in order to compute the temperature profile inside.

A measurement, which is described by the map in (0.2), is the data provided by solution generated from one non trivial initial condition $f(x,y)$. Here the main challenge is to recover $q(x)+p(y)$ from the smallest possible "number of measurements". To do so we need to exhibit special initial conditions $f(x,y)$ that would guarantee that the solution $u(x,y,t)$ carries enough spectral information about q and p accessible through the boundary. To this end we exploit the geometry of the rectangle which, as we shall see, greatly simplifies the data processing.

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