Inequalities for Fourier Transform and Convolution

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Let

$$\hat{f}(\lambda) = \int_{R^n} e^{2\pi i x.\lambda} f(x)\, dx$$

if the integral exists, be the Fourier transform of f, and let

$$(fog)(x) = \int_{R^n} f(x-y)g(y)dy$$

be the convolution of f and g. The Hausdorff-Young inequality says that if $f \in L_p(R^n)$, $1 \leq p < \infty$, and p' is the conjugate exponent of p, then $\hat{f} \in L_{p'}(R^n)$ and

$$\|\hat{f}\|_{p'} \leq C_p \|f\|_p.$$

The best constant C_p has been discovered by Babenko (1961) for even p' and Beckner (1975) for any $p > 1$.
In 1937 Pitt obtained an inequality for the Fourier transform with power weight: if $1 \leq p < q < \infty$, $0 < b < 1/p'$, $\beta = 1 - 1/p - 1/q - b < 0$, then

$$\left(\int_R |\hat{f}(\lambda)|^q |\lambda|^{\beta q} d\lambda \right)^{1/q} \leq C \left(\int_R |f(x)|^p |x|^{bp} dx \right)^{1/p}.$$

In this talk we will give an overview on the history and the latest developments in the study of inequalities for the Fourier transform with weights

$$\left(\int_{R^n} |\hat{f}(\lambda)|^q u(\lambda) d\lambda \right)^{1/q} \leq C \left(\int_{R^n} |f(x)|^p v(x) dx \right)^{1/p},$$

and also inequalities for the convolution (fog) in Lorentz spaces.